Linear Algebra

- matrix-vector notation
- solving matrix vector equations
- matrix multiplication
- matrix inverse
- transpose

$$3x = 2$$

$$x = 2/3$$

Statistics $\longrightarrow$ linear algebra

systems of equations

no x^2
x^3
x^n $n>1$
no e^x
sin

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

$$\Longrightarrow x_1 = 0, \ x_2 = 0$$

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots = b$$

In statistics, usually β is unknown

$$\begin{cases} \beta_1 + \beta_2 = 0 \\ \beta_1 - \beta_2 = 0 \end{cases}$$

$$\Rightarrow \beta_1 = 0, \ \beta_2 = 0$$

<u>Matrix-Vector Form</u>

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\substack{\text{matrix} \\ 2\times 2}} \cdot \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\substack{\text{vector} \\ 2\times 1}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\substack{\text{vector} \\ 2\times 1}}$$

$1 \cdot \beta_1 + 1 \cdot \beta_2$

$1 \cdot \beta_1 + (-1) \cdot \beta_2$

In statistics we usually have huge matrices

$\underbrace{n \times p}_{\text{dimension}}$

n = num. rows
p = num. colums

n huge
p huge

a matrix
$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix}$$

matrix X
dimension $n \times p$
defined by X_{ij} = the number at row i column j

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{X} \cdot \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{y}$$

$$X \cdot \beta = y$$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix}$$

$$X \cdot \beta = y$$

$$\underbrace{\underbrace{\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix}}_{\text{matrix} \quad n \times p} \cdot \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}}_{\substack{\text{vector} \\ p \times 1}}}_{n \times 1} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\substack{\text{vector} \\ n \times 1}}$$

"dot product"

$$\begin{bmatrix} X_{11} \cdot \beta_1 + X_{12} \cdot \beta_2 + \dots + X_{1p} \cdot \beta_p \\ X_{21} \cdot \beta_1 + X_{22} \cdot \beta_2 + \dots + X_{2p} \cdot \beta_p \\ \vdots \\ X_{n1} \cdot \beta_1 + X_{n2} \cdot \beta_2 + \dots + X_{np} \cdot \beta_p \end{bmatrix}$$

in matrix multiplication $A \cdot B$

$n_A \times p_A \quad n_B \times p_B$

we must have

~~row~~ cols A = # rows B.

$p_A = n_B$

then, the resulting matrix $A \cdot B$

$n_A \times p_B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

2×3 $\qquad$ 3×1

$$= \begin{bmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 2 \\ 4\cdot 0 + 5\cdot 1 + 6\cdot 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$$

2×1

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$$

2×3 $\qquad$ 3×2

$$= \begin{bmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 2 & 1\cdot 1 + 2\cdot 1 + 3\cdot 1 \\ 4\cdot 0 + 5\cdot 1 + 6\cdot 2 & 4\cdot 1 + 5\cdot 1 + 6\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 6 \\ 17 & 15 \end{bmatrix}$$

2×2

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$X \cdot \beta = y$$

Often times in statistics, X is a matrix of <u>known</u> numbers, y is a vector of <u>known</u> numbers, but β is usually a vector of <u>unknown</u>

number that we want to solve for.

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{cases} X_{11}\beta_1 + X_{12}\beta_2 + \dots + X_{1p}\beta_p = y_1 \\ X_{21}\beta_1 + X_{22}\beta_2 + \dots + X_{2p}\beta_p = y_2 \\ \vdots \\ X_{n1}\beta_1 + X_{n2}\beta_2 + \dots + X_{np}\beta_p = y_n \end{cases}$$

$$X \cdot \beta = y$$

Solve for β.

8th grade: $\cancel{3x=2}$

$$3 \cdot \beta = 2$$

$$3^{-1} \cdot 3\beta = 3^{-1} \cdot 2$$

multiplicative inverse
a has mult. inverse a^{-1} if $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

$$\frac{1}{3} \cdot 3\beta = \frac{1}{3} \cdot 2$$

$$1 \cdot \beta = 2/3$$

$$\beta = 2/3 \quad \square$$

now we have $X\beta = y$

<u>Want</u>

$$X^{-1} \cdot X\beta = X^{-1} y$$

hopefully

$$\beta = X^{-1} y$$

- to solve a matrix vector equation we need to understand the matrix multiplicative inverse.

8^{th} grade: $a \cdot a^{-1} = a^{-1} \cdot a = 1$

$1 \cdot \beta = \beta \cdot 1 = \beta$

Now: I is the "multiplicative identity" for any matrix $\underset{n\times p}{X}$

$$\underbrace{\underbrace{\underset{n\times p}{X}\cdot\underset{p\times \cancel{b}n}{I}}_{n\times \cancel{b}n} = \underbrace{\underset{p\times n}{I}\cdot\underset{n\times p}{X}}_{p\times p}}_{n=p} = X$$

I must be a <u>square</u> matrix $n \times n$

so, only makes sense to do

$$X \cdot I = I \cdot X = X$$ if X is also square $n \times n$.

<u>Q</u> What <u>is</u> I though?

$$\underset{n \times n}{I} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{3 \times 3}{X} \cdot \underset{3 \times 3}{I} = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = X$$

$$I \cdot X = X$$

$$\underset{1\times 1}{I} = 1$$

8th grade: $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Now: $X^{-1} \cdot X = X \cdot X^{-1} = I$

X^{-1} is defined by

$$\underset{n\times n}{X}, \quad \underset{n\times n}{I}, \quad \underset{n\times n}{X^{-1}}$$

Ex $X = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$$X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$X^{-1}X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}$$

$$XX^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I}$$

* There are good algorithms to find X^{-1} when it exists

- $\underset{n \times n}{X}$ blackbox compute X^{-1} if it exists

HW tutorial on computing matrix inverse

$\boxed{\text{Transpose}}$

In statistics, we usually have an equation like $X \cdot \beta = y$

$\underset{n \times p}{X}$ known

$\underset{n \times 1}{y}$ known

$\beta_{p \times 1}$ unknown

Want to find β.

But usually $n >> p$ and almost always $n \neq p$.

X usually not square.

So there is no X^{-1}.

trick Can we make a square invertible matrix appear?

hell yes

The transpose of a matrix $\underset{n \times p}{X}$,

is written $\underset{p \times n}{X^T}$,

$$\boxed{X^T_{ij} = X_{ji}}$$

X^T at Row i column j

$=$ X at Row j column i

$$\underset{2 \times 3}{X} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\underset{3\times 2}{X^T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

- the transpose makes a nice square (invertible) matrix appear!

$$\underset{n\times p}{X}\,\underset{p\times 1}{\beta} = \underset{n\times 1}{y}$$

$$\underbrace{\underbrace{(\underset{p\times n}{X^T}\cdot\underset{n\times p}{X})}_{p\times p}\,\underset{p\times 1}{\beta}}_{p\times 1} = \underbrace{\underset{p\times n}{X^T}\,\underset{n\times 1}{y}}_{p\times 1}$$

$$\underbrace{(X^T \cdot X)}_{p \times p} \cdot \beta = X^T \cdot y$$

$X^T X$ is square

$X^T X$ is also <u>symmetric</u>

$\underset{p \times p}{A}$ is <u>symmetric</u> means $\boxed{A_{ij} = A_{ji}}$

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 5 & 6 & 4 \end{pmatrix} \qquad A_{\substack{\text{row } i \\ \text{col } j}} = A_{\substack{\text{row } j \\ \text{col } i}}$$

$X^T X$ symmetric

$$X = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 13 \end{pmatrix}$$

$X^T X$ is square and symmetric

square $\rightarrow$ if inverse exists, we can find it

symmetric $\rightarrow$ very easy to verify if its invertable, and then to invert it

$$X \cdot \beta = y$$

$$\underbrace{(X^T X)}_{p \times p} \beta = X^T y$$

$$\underbrace{(X^T X)^{-1} \cdot (X^T X)} \cdot \beta = (X^T X)^{-1} \cdot X^T y$$

$$\underbrace{I \cdot \beta} = (X^T X)^{-1} X^T y$$

$$\underset{p \times 1}{\beta} = \underbrace{(X^T X)^{-1} X^T y}_{p \times 1}$$

$$\underbrace{\underbrace{(\underset{p \times n}{X^T} \underset{n \times p}{X})^{-1}}_{p \times p} \; \underbrace{\underset{p \times n}{X^T} \; \underset{n \times 1}{y}}_{p \times 1}}_{p \times 1}$$

$$X\beta = y$$

$$X^T X \beta = X^T y$$

$$\beta = (X^T X)^{-1} X^T y$$

Online, math 2400, i'll give a private lesson

- how to compute matrix inverse
- why symmetric square matrices are so wonderful

end of class, might have its own lecture

What it means to do Research

Reading

<u>Step 1</u> Ask a question(s)!

why is $e^{i\pi} = 1$?

why did I not go to NBA?

What makes someone go to the NBA?

<u>Step</u> Is it a <u>good research</u> question for me?

- is it the right level of difficulty for me?
- Can I answer it?
- has it been answered before?
- will anyone give a shit?
- is it quantifiable?

— do you care enough about this question to put in the time?

Research is a series of incremental improvements in answering a question.

Step 3 Literature Review

— related work

— find your starting point by reading

Step 4 Remake wins above replacement for pitchers

— write down the equation