Kelly Betting

Want to bet on basketball games over the course of an entire NBA season and beyond.

Initial bankroll $B = \$100$

P_i = true prob. that team A_i wins against team B_i (assume known)

decimal odds α_{A_i} = # dollars returned for a \$1 bet on team A_i

= 1 + profit from a \$1 bet on A_i if it hits

<u>Q</u> How should we bet?

Maximize EV

EV of a \$1 bet on team A:

$$EV_A = \mathbb{E}\left(\text{Profit if bet on } A\right) = \left(\begin{matrix}\text{Prob.}\\ A \text{ wins}\end{matrix}\right)\cdot\left(\begin{matrix}\text{Profit}\\ \text{if } A\\ \text{wins}\end{matrix}\right) + \left(\begin{matrix}\text{Prob.}\\ A\\ \text{loses}\end{matrix}\right)\cdot\left(\begin{matrix}\text{Profit}\\ \text{if}\\ A\\ \text{loses}\end{matrix}\right)$$

$$= P(\alpha_A - 1) + (1-P)(-1) = P\alpha_A - 1$$

Traditional approach:

If $EV_A < 0$, on average you'll lose money, don't bet

If $EV_A > 0$, on average you'll profit, bet

But what if $EV_A > 0$ and you bet your entire bankroll?

You will be depressed if you lose.

How do we rectify this?

Supposing each game is +EV, bet $\frac{B}{N}$ on each and that will be +EV and also you're much less likely to be depressed.

Makes money on average.

Can we do better?

Can we take advantage of the sequential nature of the bets? Compounding: if I make money on the first bet, I'll want to use that money to bet in the next game.

Bet a fraction $f \in [0, 1]$ of our bankroll $\$B$.

Bet size $B \cdot f_1$ on team A_1 in game 1

$$\text{Profit} = \begin{cases} +(\alpha_{A_1} - 1) \cdot B f_1 & \text{w.p. } p \quad (A_1 \text{ wins}) \\ -B \cdot f_1 & \text{w.p. } 1-p \quad (A_1 \text{ loses}) \end{cases}$$

$$= B \cdot f_1 \cdot (\alpha_{A_1} X_1 - 1)$$

$$X_1 = \begin{cases} 1 & \text{if } A_1 \text{ wins (w.p. } p) \\ 0 & \text{if } A_1 \text{ loses (w.p. } 1-p) \end{cases}$$

Profit after 1st bet

What do we want to maximize?

↳ Maximize Bankroll after N bets,

After 1 bet: $B + B \cdot f_1 \cdot (\alpha_{A_1} X_1 - 1)$

$$= B[1 + f_1(\alpha_{A_1} X_1 - 1)]$$

After 2 bets:

initial bankroll

$$B[1 + f_1(\alpha_{A_1} X_1 - 1)]$$

$$+ B[1 + f_1(\alpha_{A_1} X_1 - 1)] \cdot f_2(\alpha_{A_2} X_2 - 1)$$

profit

$$= B[1 + f_1(\alpha_{A_1} X_1 - 1)][1 + f_2(\alpha_{A_2} X_2 - 1)]$$

After N bets:

B = initial bankroll
f_i = fraction of bankroll bet on team A_i
α_{A_i} = decimal odds = 1 + profit if hit \$1 bet on A_i
X_i = game outcome = 1 if A_i wins else 0

$$\boxed{\text{Bankroll}_N = B \prod_{i=1}^{N} \{1 + f_i(\alpha_{A_i} X_i - 1)\}}$$

$$= B(1 + f_1(\alpha_{A_1} X_1 - 1)) \cdot (1 + f_2(\alpha_{A_2} X_2 - 1)) \cdot \ldots$$

Known fixed constant

B = initial bankroll

want to find

f_i = fraction of bankroll bet on team A_i

α_{A_i} = decimal odds = 1+ profit if hit \$1 bet on A_i

random variable

X_i = game outcome = 1 if A_i wins else 0

$$\boxed{\text{Bankroll}_N = B \prod_{i=1}^{N} \left\{1 + f_i(\alpha_{A_i} X_i - 1)\right\}}$$

Bankroll_N is a random variable

Want to maximize expected bankroll. $\rightarrow$ a number

$$\underset{f=(f_1,\dots,f_N)}{\operatorname{argmax}} \ \mathbb{E}[\text{Bankroll}_N]$$

$$= \underset{f}{\operatorname{argmax}} \ \mathbb{E}\left[B \prod_{i=1}^{N} \left\{1 + f_i(\alpha_{A_i} X_i - 1)\right\}\right]$$

Good luck.

$$= \underset{f}{\operatorname{argmax}} \ \log\left(\mathbb{E}\left[B \prod_{i=1}^{N} \left\{1 + f_i(\alpha_{A_i} X_i - 1)\right\}\right]\right)$$

since log is monotonic increasing

$\log \mathbb{E}\, g(X) \neq \mathbb{E} \log g(X)$ for most g's

Kelly (Eitan): Who cares just do it anyways

Instead, find

$$\underset{f}{\operatorname{argmax}} \; \mathbb{E}\left[\log\left(B \prod_{i=1}^{N} \{1 + f_i(\alpha_{A_i} X_i - 1)\}\right)\right]$$

Shannon-McMillan-Breiman Thm 1950s:
the f that maximizes expected log bankroll produces more money asymptotically as $N \to \infty$ than any other allocation f_i'

$$= \underset{f}{\operatorname{argmax}} \; \mathbb{E}\left[\cancel{\log(B)} + \sum_{i=1}^{n} \log\{1 + f_i(\alpha_{A_i} X_i - 1)\}\right]$$

$$= \underset{f}{\operatorname{argmax}} \sum_{i=1}^{n} \mathbb{E} \log(1 + f_i(\alpha_{A_i} X_i - 1))$$

$X_i = \begin{cases} 1 & \text{w.p. } p \text{ if } A_i \text{ wins} \\ \text{else } 0 \end{cases}$

$$= \underset{(f_1, \ldots, f_N)}{\operatorname{argmax}} \sum_{i=1}^{n} \left[p \cdot \log(1 + f_i(\alpha_{A_i} - 1)) + (1-p) \cdot \log(1 - f_i)\right]$$

$$\downarrow$$

$$\underset{f_i}{\operatorname{argmax}} \quad p \cdot \log\left(1 + f_i(\alpha_{A_i} - 1)\right) + (1-p) \cdot \log(1 - f_i)$$

Calculus.

$$0 = \frac{d}{df_i}\left[p \cdot \log\left(1 + f_i(\alpha_{A_i} - 1)\right) + (1-p) \cdot \log(1 - f_i)\right]$$

$$0 = \frac{\alpha - 1}{1 + f(\alpha - 1)} \cdot p - (1-p) \cdot \frac{1}{1 - f}$$

$$\frac{1-p}{1-f} = \frac{p(\alpha - 1)}{1 + f(\alpha - 1)}$$

$$(1-p)(1 + f(\alpha - 1)) = (1 - f)(p(\alpha - 1))$$

$$f(1-p)(\alpha - 1) + (1-p) = -fp(\alpha - 1) + p(\alpha - 1)$$

$$f\left[(1-p)(\alpha - 1) + p(\alpha - 1)\right] = p(\alpha - 1) - (1-p) = p\alpha - p - 1 + p = p\alpha - 1$$

$$f = \frac{p\alpha - 1}{\alpha - 1} \longrightarrow \boxed{f_i = \max\left(0, \frac{p_i \alpha_{A_i} - 1}{\alpha_{A_i} - 1}\right)}$$

Kelly Fraction

- the fraction of your bankroll you should bet on game i to maximize expected log wealth assuming true wp is known.

exs

- If $p=1$ (guaranteed to hit the bet), $f=1$ (bet entire bankroll)
- If $p<1$, then $\frac{p\alpha-1}{\alpha-1} < \frac{1\cdot\alpha-1}{\alpha-1} = 1$
  if not guaranteed to win, then don't bet all your bankroll
- If $p=0$, $f=0$
- If the odds are fair, $\alpha = \frac{1}{p}$ $\left(\text{e.g. } p=\frac{1}{2},\ \alpha = 2 = 1+1\right)$
  [α = 1 + profit of a \$1 bet if it hits]

  and $\frac{p\alpha-1}{\alpha-1} = \frac{p\left(\frac{1}{p}\right)-1}{\frac{1}{p}-1} = 0$
- If you have an edge $\delta>0$, $\alpha = \frac{1}{p}+\delta$, then

$$\frac{p\alpha-1}{\alpha-1} = \frac{p\left(\frac{1}{p}+\delta\right)-1}{\left(\frac{1}{p}+\delta\right)-1} = \left[\frac{\delta}{\frac{1}{p}-1+\delta}\right]\frac{\left(\frac{1}{\delta}\right)}{\left(\frac{1}{\delta}\right)} = \frac{1}{1+\frac{1/p-1}{\delta}}$$

as edge $\delta \uparrow$, kelly fraction $f \uparrow$

"bet your edge"

In practice, the win probability p of a team winning a game is unknown/unobservable.
It needs to be estimated from data, $\hat{p}$.
How does kelly betting change under this regime?

Ideally the estimator $\hat{p}$ is unbiased $\mathbb{E}\hat{p} = p$
but subject to some uncertainty $\text{Var}(\hat{p}) = \tau^2$
The more uncertain we are in our estimate
the less we should bet.

Fractional Kelly says bet a fraction $k \in [0,1]$
of the kelly fraction f, $f \leftarrow f \cdot k$

"Half kelly" "Quarter kelly"