Kelly Betting

Card game demo

- split up into pairs
- each pair grab a deck of cards
- each person starts with, say, $100
- let person A in the pair be the dealer and person B be the bettor.

Person B bets as much as he wants of his bankroll on whether the next card will be Red or Black at ~~Fair Odds~~ Even odds. Person A flips a card. Record the new bankroll after the card is flipped. Play until 35 cards have been flipped.

- Then Reverse the roles A and B and replay
- The person who makes the most money wins!

We want to bet on Basketball games over the course of the entire NBA season and beyond.

Initial Bankroll $B = \$100$.

P_i = true Prob. that team A_i wins (assume known) vs. team B_i in game i

Decimal odds α_{A_i} = # dollars returned for a \$1 bet on team A_i
$= 1 +$ profit from \$1 bet on team A_i

α_{B_i}

Q How should we bet?

* Someone's going to say: maximize EV

* EV of a \$1 bet on team A: ignoring subscript i,

$$EV_A = \mathbb{E}(\text{profit}_A) = \begin{pmatrix}\text{pr. } A \\ \text{wins}\end{pmatrix} \cdot \begin{pmatrix}\text{profit} \\ \text{if } A \\ \text{wins}\end{pmatrix} + \begin{pmatrix}\text{pr. } A \\ \text{loses}\end{pmatrix} \cdot \begin{pmatrix}\text{profit} \\ \text{if } A \\ \text{loses}\end{pmatrix}$$

$$= p(\alpha_A - 1) + (1-p)(-1)$$

$$= p\alpha_A - 1$$

$$\mathbb{E}(\text{profit}_B) = (1-p)\alpha_B - 1$$

* Let's say $EV_A < 0$. How much will you bet?

→ they'll say ZERO. Fair enough.
Come back to this

* Let's say $EV_A > 0$. How much will you bet?

→ they'll say make a bet!

* Even if $EV_A > 0$, if you bet you're entire bankroll on A and lose, then you're <u>out</u> <u>of</u> <u>money</u> :(

How to account for this?

→ What about, over N bets, just split up you're money evenly betting B/N on team A each time?

→ Make money on average
but can we make more money?
What are we missing?

* Not taking advantage of the sequential nature of the bets.
If I make money on the first bet, I can use that profit to bet more on the second bet! compounding!

* How do we actually acheive compounding?
$\longrightarrow$ bet a fraction $f \in (0,1)$ of your bankroll

Bet size $B \cdot f_1$ on team A_1 in game 1
Make $(\alpha_{A_1} - 1) B f_1$ w.p. p
~~Lose~~ make $-B f_1$ w.p. $1-p$
Profit $B \cdot f_1 (\alpha_{A_1} X_1 - 1)$
where $X_1 = \begin{cases} 1 \text{ if } A_1 \text{ wins in game 1} \\ 0 \text{ if } A_1 \text{ loses in game 1} \end{cases}$

* Is this what we want to maximize though? Profit? What do we actually want to have at the end?
A high bankroll!

Bankroll $B + Bf_1(\alpha_{A_1}X_1 - 1) = B[1 + f_1(\alpha_{A_1}X_1 - 1)]$

$\downarrow$

After first bet, we have $B[1 + f_1(\alpha_{A_1}X_1 - 1)]$

Bet size $B[1 + f_1(\alpha_{A_1}X_1 - 1)] \cdot f_2$ on tm A_2 in game 2

Profit $B[1 + f_1(\alpha_{A_1}X_1 - 1)] \cdot f_2(\alpha_{A_2}X_2 - 1)$ after game 2

Bankroll $B[1 + f_1(\alpha_{A_1}X_1 - 1)][1 + f_2(\alpha_{A_2}X_2 - 1)]$ by same logic

$\downarrow$

After N games,

$$\boxed{\text{Bankroll} = B \prod_{i=1}^{N} \left[1 + f_i(\alpha_{A_i}X_i - 1)\right]}$$

B = initial bankroll (say, \$100)

α_{A_i} = decimal odds for betting on team A in game i (known)

$X_i = 1$ if team A wins in game i, else 0 (random variable)

f_i = fraction of bankroll on game i (want to find)

* Want to maximize Bankroll

but, bankroll is a Random Variable

$\longrightarrow$ Maximize Expected Bankroll (a number)

$$\underset{f}{\operatorname{argmax}} \; \mathbb{E} \text{ BankRoll}$$

$$= \underset{f}{\operatorname{argmax}} \; \mathbb{E}\left(B \prod_{i=1}^{N} \left[1 + f_i (\alpha_{A_i} X_i - 1) \right] \right)$$

the random variables $X_1, \dots, X_N$

* Good luck doing this!!

Too hard due to the product.

So, we're stuck because of the product.

How to get rid of a product?

$\longrightarrow$ log

Kelly's brilliant idea:

Try $$\operatorname*{argmax}_f \mathbb{E} \log \left(B \prod_{i=1}^{N} \left[1 + f_i (\alpha_{A_i} X_i - 1) \right] \right)$$

Shannon-McMillan-Breiman 1950s:

the f that maximizes the log bankroll has more money asymptotically as N goes to ∞ than any other allocation f!

$$= \operatorname*{argmax}_f \mathbb{E} \sum_{i=1}^{n} \log \left(1 + f_i (\alpha_{A_i} X_i - 1) \right)$$

$$= \sum \mathbb{E} \log$$

$$= \operatorname*{argmin}_f \sum_{i=1}^{n} \log \left(1 + f_i (\alpha_{A_i} - 1) \right) \cdot P_i + \log (1 - f_i) \cdot (1 - P_i)$$

Same minimization for each i, due to the Σ

$$\underset{f}{\text{argmin}} \quad \log(1+f(\alpha-1))p + \log(1-f)(1-p)$$

* How to solve this? Calculus!

$$\frac{d}{df}\left[\log(1+f(\alpha-1))p + \log(1-f)(1-p)\right]$$

$$= \frac{\alpha-1}{1+f(\alpha-1)}\cdot p + \frac{-1}{1-f}\cdot(1-p) = 0$$

$$\Rightarrow \frac{1-p}{1-f} = \frac{p(\alpha-1)}{1+f(\alpha-1)}$$

$$\Rightarrow (1-p)(1+f(\alpha-1)) = p(\alpha-1)(1-f)$$

$$\Rightarrow f(1-p)(\alpha-1) + (1-p) = -fp(\alpha-1) + p(\alpha-1)$$

$$\Rightarrow f((1-p)(\alpha-1) + p(\alpha-1)) = p(\alpha-1) - (1-p)$$

$$\Rightarrow f(\alpha-1) = p\alpha \cancel{-p} - 1 \cancel{+p}$$

$$\Rightarrow f = \frac{p\alpha-1}{\alpha-1} \qquad \boxed{f = \max\left(0, \frac{p\alpha-1}{\alpha-1}\right)}$$

Kelly Fraction

Exs

- If $p=1$ (guaranteed)
  then $f=1$ (bet entire bankroll)

- If -110 bet,
  decimal odds $\alpha = 1 + \frac{100}{110} = \frac{210}{110} = 1.909$
  for f to be positive (to bet something)
  we need $\frac{p\alpha - 1}{\alpha - 1} > 0 \Rightarrow p > \frac{1}{\alpha} = 0.524$

- Fair odds is $\alpha = \frac{1}{p}$
  Suppose $\alpha = \frac{1}{p} + \delta$, so $\delta \geq 0$ is your "edge"
  Then $\frac{p\alpha - 1}{\alpha - 1} = \frac{p\left(\frac{1}{p} + \delta\right) - 1}{\frac{1}{p} + \delta - 1} = \frac{\delta}{\delta + \frac{1}{p} - 1} = \frac{1}{1 + \frac{1/p - 1}{\delta}}$
  Bet your edge: as $\delta \uparrow$, $f \uparrow$

- Desmos $f = \max\left(0, \frac{p\alpha - 1}{\alpha - 1}\right)$

* In practice, the win probability p of a horse or team is not an observable or known quantity (with a deck of cards it is, but in real life sports it's not); it needs to be estimated from data $\rightarrow \hat{p}$.

How does Kelly betting change under this?

* Ideally our estimator $\hat{p}$ of p is unbiased $\mathbb{E}\hat{p} = p$ but subject to some uncertainty $\mathrm{Var}(\hat{p}) = \tau^2$.

The more uncertain we are in our estimate, the less we should bet.

* Fractional Kelly says bet a fraction $K \in [0,1]$ of the Kelly betting fraction f, $f \leftarrow K \cdot f$.

* K is some function $K = K(\tau)$ such that $\lim_{\tau \downarrow 0} K(\tau) = 1$

(if $\mathbb{E}\hat{p} = p$ and $\mathrm{var}(\hat{p}) = 0$, we know the true win probabilities, so use original Kelly fraction)

and $\lim_{\tau \uparrow \infty} K(\tau) = 0$

(fully uncertain about win prob)

* K-Fractional Kelly with $\hat{p}$ is equivalent to full Kelly with a shrinkage estimator for $\hat{p}$, shrinking more if τ^2 larger