Estimating the Probability of a Rare Event

Q LeBron James has played about 1,750 games in his NBA career.
He averages 27 points, 7 rebounds, and 7 assists (27-7-7) but has not once had a game with this box score.
How unlikely is this? How crazy is this?

First thought: pretty crazy?? Right??

Rough estimate of the probability he hits 27-7-7 in <u>one</u> game:
assuming pts, reb, and assists are independent,

say $\mathbb{P}(27 \text{ pts}) = 1/20$ 5%
say $\mathbb{P}(7 \text{ reb}) = 1/10$ 10%
say $\mathbb{P}(7 \text{ assists}) = 1/10$ 10%

$p = \mathbb{P}(27\text{-}7\text{-}7) = 1/2000$ a rare event!

How about the likelihood he doesn't hit 27-7-7 across <u>any</u> of n games?

Model $X \sim \text{Binom}(n, p)$

$n = 1750$
$p \approx 1/2000$

$$\mathbb{P}\left(\#\ \text{27-7-7 games in } n=1750\right) = \mathbb{P}(X=0)$$

$$= \binom{n}{0} p^0 (1-p)^n$$

$$= (1-p)^n$$

$$= \left(\frac{1999}{2000}\right)^n$$

way too hard to compute casually courtside with your buddies

Need a better way to estimate this!

Law of Rare Events / Poisson Limit Theorem

Suppose $X \sim \text{Binom}(n, p_n)$ where $\lim_{n\to\infty} np_n = \lambda \in (0,\infty)$. Then $X \xrightarrow{d} \text{Poisson}(\lambda)$.

Proof Idea

$$\mathbb{P}(X=k) = \frac{n!}{k!(n-k)!} p_n^k (1-p_n)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \frac{(np_n)^k}{n^k} \left(1-\frac{np_n}{n}\right)^{n-k}$$

$$= \frac{n!}{n^k(n-k)!} \frac{(np_n)^k}{k!} \left(1-\frac{np_n}{n}\right)^n \left(1-\frac{np_n}{n}\right)^{-k}$$

as $n\to\infty$, $\displaystyle\frac{n!}{n^k(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n\cdot n\cdot n\cdots n}\cdot\frac{\cancel{(n-k)!}}{\cancel{(n-k)!}} \longrightarrow 1$

as $n\to\infty$, $\displaystyle\left(1-\frac{np_n}{n}\right)^{-k} \approx \left(1-\frac{\lambda}{n}\right)^{-k} \longrightarrow 1$

as $n\to\infty$, $\displaystyle\left(1-\frac{np_n}{n}\right)^{n} \approx \left(1-\frac{\lambda}{n}\right)^{n} \longrightarrow e^{-\lambda}$

as $n\to\infty$, $\displaystyle\frac{(np_n)^k}{k!} \approx \frac{\lambda^k}{k!}$

So as $n\to\infty$, $\displaystyle\mathbb{P}(X=k) \approx e^{-\lambda}\frac{\lambda^k}{k!} = \mathbb{P}(\text{Poisson}(\lambda)=k)$ $\square$

$n = 1750 \approx 2000$ opportunities

$$\lambda = np \approx 2000 \cdot \frac{1}{2000} = 1$$

$$\mathbb{P}\left(\begin{matrix}\text{no 27-7-7 games} \\ \text{in } n \text{ tries}\end{matrix}\right) \approx \mathbb{P}(\text{Poisson}(\lambda) = 0)$$

$$= e^{-\lambda} \approx \frac{1}{2.71} \approx 37\%$$

This is a <u>super rough</u> approximation but it's Not crazy that we haven't seen it !!

Impress your friends with the e^{-np} trick!