Understanding the Nonparametric Empirical Bayes Estimator for Brown 2008

Model

$$\boxed{\begin{cases} X_i \mid \theta_i \overset{ind}{\sim} N(\theta_i, \sigma_i^2) \\ \theta_i \overset{ind}{\sim} \underbrace{G}_{\text{some CDF}} \end{cases}}$$

σ_i^2 known

θ_i, G unknown

Posterior

$$P(\theta_i \mid X_i) = \frac{P(X_i \mid \theta_i)\, P(\theta_i)}{P(X_i)} \qquad \text{Bayes Rule}$$

$$= \frac{P(X_i \mid \theta_i)\, P(\theta_i)}{\int P(X_i \mid \theta_i)\, P(\theta_i)\, d\theta_i}$$

$$= \frac{\phi\left(\frac{X_i - \theta_i}{\sigma_i}\right) dG(\theta_i)}{\int \phi\left(\frac{X_i - \theta_i}{\sigma_i}\right) dG(\theta_i)}$$

ϕ = std normal density

Bayes Estimator: Posterior Mean

$$\hat{\theta}_i^{(Bayes, G)} = \mathbb{E}(\theta_i \mid X_i) = \int \theta_i\, P(\theta_i \mid X_i)\, d\theta_i$$

$$\boxed{\hat{\theta}_i^{(G)}(X_i) = \frac{\int \theta_i\, \phi\left(\frac{X_i - \theta_i}{\sigma_i}\right) dG(\theta_i)}{\int \phi\left(\frac{X_i - \theta_i}{\sigma_i}\right) dG(\theta_i)}}$$

Problem G is unknown, so we can't directly compute this integral

Lemma 1

$$\boxed{\hat{\theta}_i^{(G)}(X_i) = X_i + \sigma_i^2 \left[\frac{\frac{\partial g_i^*}{\partial x}(X_i)}{g_i^*(X_i)} \right]}$$

marginal density of X_i ← where $g_i^*(X_i) = \int \phi\left(\frac{X_i - \theta_i}{\sigma_i}\right) dG(\theta_i)$ [redacted]

Pf Remove the "i" for notational convenience.

$$\sigma^2 \left(\frac{\frac{\partial g^*}{\partial x}(x)}{g^*(x)} \right) = \sigma^2 \frac{\int \left[\frac{\partial}{\partial x} \phi\left(\frac{x-\theta}{\sigma}\right)\right] dG(\theta)}{\int \phi\left(\frac{x-\theta}{\sigma}\right) dG(\theta)}$$

$$= \frac{\sigma^2 \int \left(-\left(\frac{x-\theta}{\sigma}\right)\right) \phi\left(\frac{x-\theta}{\sigma}\right) \cdot \left(\frac{1}{\sigma}\right) dG(\theta)}{\int \phi\left(\frac{x-\theta}{\sigma}\right) dG(\theta)}$$

$$= \frac{\int (\theta - x)\, \phi\left(\frac{x-\theta}{\sigma}\right) dG(\theta)}{\int \phi\left(\frac{x-\theta}{\sigma}\right) dG(\theta)}$$

$$= \hat{\theta}^{(G)} - X$$

$\square$

Note

$$\frac{d}{dy}\phi(y) = \frac{d}{dy} \frac{e^{-y^2/2}}{\sqrt{2\pi\sigma^2}}$$

$$= \frac{e^{-y^2/2}}{\sqrt{2\pi\sigma^2}}(-y)$$

$$= -y\,\phi(y)$$

Problem Because G is unknown, g^* is unknown, so we need to estimate g^* as $\tilde{g}^*$ to get the

Nonparametric Empirical Bayes Estimator $\boxed{\hat{\theta}_i^{(NPEB)}(X) = X_i + \sigma_i^2 \frac{\frac{\partial \tilde{g}_i^*}{\partial x_i}(X_i)}{\tilde{g}_i^*(X_i)}}$

Fix $x \in \mathbb{R}$.

Def $A_{ik}(x) := \phi\left(\frac{x-X_k}{\sqrt{(1+h)(\sigma_k^2 \vee \sigma_i^2)-\sigma_k^2}}\right)\frac{1}{\sqrt{(1+h)(\sigma_k^2 \vee \sigma_i^2)-\sigma_k^2}}$

Def $I_{ik}(x) := \int \phi\left(\frac{x-\theta}{\sqrt{(1+h)(\sigma_k^2 \vee \sigma_i^2)}}\right)\frac{1}{\sqrt{(1+h)(\sigma_k^2 \vee \sigma_i^2)}}\, G(d\theta)$

Def Let $h>0$ small. Define $\mathcal{K} := \{k : \sigma_k^2 \le (1+h)\sigma_i^2\}$

Lemma $\mathbb{E} A_{ik}(x) = I_{ik}(x)$

Pf ??

Conclusion If $k \in \mathcal{K}$, then $(1+h)(\sigma_k^2 \vee \sigma_i^2) = (1+h)\max\{\sigma_k^2, \sigma_i^2\}$

$$\le (1+h)^2 \sigma_i^2 \approx \sigma_i^2,$$

so $$I_{ik} \approx \int \phi\left(\frac{x-\theta}{\sigma_i}\right)\frac{1}{\sigma_i} G(d\theta) = \frac{1}{\sigma_i} g_i^*(x).$$

So to estimate $g_i^*(x)$, we need to estimate I_{ik} for $k \in \mathcal{K}$.
To do so, we simply take the average over all of A_{ik} $k \in \mathcal{K}$.

Def $I_i := \frac{1}{|\mathcal{K}|}\sum_{k \in \mathcal{K}} A_{ik}(X_i)$

Def $\widehat{g_i^*}(X_i) := \sigma_i I_i$