Understanding Parametric Empirical Bayes, as used in Brown 2008

Model $\begin{cases} X_i \mid \theta_i \sim N(\theta_i, \sigma_i^2) \\ \theta_i \sim N(\mu, \tau^2) \end{cases}$ σ_i^2 known

θ_i, μ, τ^2 unknown

→ the PRIOR has a specified distributional form ⟹ "Parametric"

Posterior $\theta_i \mid X_i \sim N\left(\dfrac{\frac{X_i}{\sigma_i^2} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma_i^2} + \frac{1}{\tau^2}}, \dfrac{1}{\frac{1}{\sigma_i^2} + \frac{1}{\tau^2}} \right)$ Bayes Rule & Complete the Square

Bayes Estimator: Posterior Mean $\hat{\theta}_i^{(Bayes)} = \mathbb{E}(\theta_i \mid X_i) = \dfrac{\frac{X_i}{\sigma_i^2} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma_i^2} + \frac{1}{\tau^2}}$

$$\Rightarrow \boxed{\hat{\theta}_i^{(Bayes)} = \frac{\tau^2}{\tau^2 + \sigma_i^2}(X_i - \mu) + \mu}$$

Problem μ, τ^2 are unknown, so we can't use $\hat{\theta}_i^{(Bayes)}$ directly

Solution: Empirical Bayes Estimate the Bayesian hyperparameters μ, τ^2 from the data $\{X_i\}$, and use these estimates $\hat{\mu}, \hat{\tau}^2$ in place of μ, τ^2 in $\hat{\theta}_i^{(Bayes)}$

① Parametric Empirical Bayes (Maximum Likelihood)

Idea Use $\hat{\mu} = \hat{\mu}_{MLE}$, $\hat{\tau}^2 = \hat{\tau}^2_{MLE}$ as our estimators for μ, τ^2

[redacted] from the data $\{X_i\}$

Marginal $X_i \sim N(\mu, \tau^2 + \sigma_i^2)$

i^{th} Likelihood $P(X_i) = \dfrac{1}{\sqrt{2\pi(\tau^2 + \sigma_i^2)}} e^{-\frac{(X_i - \mu)^2}{2(\tau^2 + \sigma_i^2)}}$

Full Log Likelihood $\ell(\vec{x} \mid \mu, \tau^2, \vec{\sigma}^2) = \sum_{i=1}^{p} \log P(X_i)$

$$= -\frac{1}{2} \sum_{i=1}^{p} \frac{(X_i - \mu)^2}{(\tau^2 + \sigma_i^2)} - \frac{1}{2} \sum_{i=1}^{p} \log(\tau^2 + \sigma_i^2) + C$$

Find $\hat{\mu}_{MLE}$ $\quad \dfrac{\partial \ell}{\partial \mu} = \displaystyle\sum_{i=1}^{p} \frac{(X_i-\mu)}{\tau^2+\sigma_i^2}$

$$\frac{\partial \ell}{\partial \mu}(\hat{\mu}) = 0 \implies \sum_{i=1}^{p} \frac{X_i}{\tau^2+\sigma_i^2} = \sum_{i=1}^{p} \frac{\hat{\mu}}{\tau^2+\sigma_i^2}$$

$$\implies \hat{\mu} = \frac{\sum X_i/(\tau^2+\sigma_i^2)}{\sum 1/(\tau^2+\sigma_i^2)}$$

Find $\hat{\tau}^2_{MLE}$ $\quad \dfrac{\partial \ell}{\partial \tau^2} = \dfrac{1}{2} \displaystyle\sum_{i=1}^{p} \frac{(X_i-\mu)^2}{(\tau^2+\sigma_i^2)^2} - \frac{1}{2} \sum_{i=1}^{p} \frac{1}{\tau^2+\sigma_i^2}$

$$\frac{\partial \ell}{\partial \tau^2}(\hat{\tau}^2) = 0 \implies \sum \frac{(X_i-\mu)^2}{(\hat{\tau}^2+\sigma_i^2)^2} = \sum \frac{1}{(\hat{\tau}^2+\sigma_i^2)}$$

Solve this system to get $\hat{\tau}^2$.

Now, $\hat{\mu}$ is in terms of τ^2, and $\hat{\tau}^2$ is in terms of μ.

Hence, to obtain $\hat{\mu}_{MLE}$ and $\hat{\tau}^2_{MLE}$, we

$$\boxed{\begin{gathered} \text{Iteratively Solve for } \hat{\mu}, \hat{\tau}^2: \\ \hat{\mu} = \frac{\sum X_i/(\hat{\tau}^2+\sigma_i^2)}{\sum 1/(\hat{\tau}^2+\sigma_i^2)} \\ \sum \frac{(X_i-\hat{\mu})^2}{(\hat{\tau}^2+\sigma_i^2)^2} = \sum \frac{1}{\hat{\tau}^2+\sigma_i^2} \end{gathered}}$$

Then, once we've obtained $\hat{\mu}, \hat{\tau}^2$ via MLE, the final Parametric Empirical Bayes estimator for θ is

$$\boxed{\hat{\theta}_i^{(EBML)} = \frac{\hat{\tau}^2}{\hat{\tau}^2+\sigma_i^2}(X_i-\hat{\mu}) + \hat{\mu}}$$