Kelly Criterion. Case: No Rake/ Vigorish/ "Track Take"

Setup

- Consider a series of upcoming events that you would like to bet on. For instance, a series of N baseball games.

- For each event, suppose there are n outcomes

- Suppose you have an "information channel" that sends you a predicted outcome $r \in \{1, \ldots, n\}$ for each event/game. This info channel can be a real communication channel or represent the totality of inside information available to the gambler.

- We want to know how much money we can make by betting, using the info from the Info Channel

Notation

α_s = odds paid if s^{th} outcome occurs
= # of dollars returned for a \$1 bet.

$p(s)$ = probability that s^{th} outcome occurs.

$a(s/r)$ = the fraction of the gambler's capital that she decides to bet on outcome s, given that she receives symbol r = our Gambling Strategy

V_N = gambler's capital after N bets
V_0 = gambler's initial capital
W_{rs} = the # of times you receive symbol r, and the outcome is s, in N games
G = exponential rate of growth of gambler's capital
$P(s,r)$ = probability that outcome s occurs, and receive symbol r
$q(r)$ = probability you receive symbol r
$q(s|r)$ = probability outcome s occurs, given that you receive r

1. case: Fair odds, No Rake/Vigorish

Fair odds

$$\boxed{\alpha_s = \frac{1}{p(s)}}$$

α_s = odds paid if outcome s occurs

$p(s)$ = probability outcome s occurs

"Fair" because if you bet \$B that outcome s will occur, then you're expected profit it

$$\mathbb{E}[\text{Profit}] = p(s)\cdot(\alpha_s B - B) + (1-p(s))(-B)$$
$$= B - p(s)\cdot B - B + p(s)\cdot B$$
$$= 0.$$

Fact

$$\boxed{1 = \sum \frac{1}{\alpha_s}}$$

Without loss of generality, impose the

Constraint

$$\boxed{1 = \sum_s a(s|r) \quad \forall r}$$

$a(s|r)$ = the fraction of gambler's capital bet on outcome s, given that receives symbol r = our gambling strategy

"Regardless of the symbol r the gambler receives, she will bet her entire capital"

We can assume this because the gambler can hold back some capital by placing cancelling bets; because the odds are fair, there is no rake to make cancelling bets unprofitable.

$$\boxed{V_N = \prod_{r,s} \left[a(s|r) \cdot \alpha_s\right]^{W_{rs}} \cdot V_0}$$

V_N = gambler's capital after N bets.
V_0 = gambler's initial capital.
W_{rs} = the # of times that you receive symbol r, and the outcome is s, in N games.

$$\boxed{G := \lim_{N \to \infty} \frac{1}{N} \log_2\left(\frac{V_N}{V_0}\right)}$$

G = Exponential rate of growth of gambler's capital

$$V_N \approx V_0 (e^G)^N$$

Goal Maximize G (maximize the Logarithm of the gambler's capital)

Why G? - logarithm makes maximization easy

- Maximize accumulated/compound capital V_N instead of maximizing (expected) capital of each bet
- If you look at each bet individually, you put all your money on the most likely outcome, which if it doesn't happen means you bust...

$$G = \lim_{N\to\infty} \frac{1}{N} \log_2\left(\frac{V_N}{V_0}\right)$$

$$= \lim_{N\to\infty} \sum_{rs} \left(\frac{W_{rs}}{N}\right) \log\left[\alpha_s\, a(s|r)\right]$$

$$= \sum_{rs} p(s,r) \log\left[\alpha_s\, a(s|r)\right]$$

P(s,r) = Probability that outcome s occurs and receive symbol r

$$= \sum_{rs} p(s,r) \log\left[\frac{\alpha(s|r)}{p(s)}\right]$$

$$= \sum_{rs} p(s,r) \log a(s|r) - \underbrace{\sum_{rs} p(s,r) \log p(s)}_{\text{Shannon's "Source Rate" } H(X)}$$

Goal Maximize G.

p(s,r) is fixed, but unknown (in practice, estimate it, but here, assume known).

a(s|r) is the Gambling Strategy, which the gambler has control over! The gambler chooses her strategy!

So, how can we choose the gambling strategy a(s|r) to maximize G?

Compute

$$\underset{a}{\operatorname{argmax}}\, G = \underset{a}{\operatorname{argmax}} \sum_{rs} p(s,r) \log a(s|r)$$

$$= \underset{a}{\operatorname{argmax}} \sum_{rs} q(r)\, q(s|r) \log a(s|r)$$

q(r) = probability that you receive symbol r

q(s|r) = probability that outcome s occurs given that you received symbol r

$$= \underset{a}{\operatorname{argmax}} \sum_r q(r) \sum_s q(s|r) \log a(s|r)$$

Equivalently, $\forall r$, $\underset{a}{\operatorname{argmax}} \sum_s q(s|r) \log a(s|r)$

- For notational convenience, we may drop the r and write $\underset{a}{\operatorname{argmax}} \sum_s q(s) \log a(s)$, knowing there is an implicit conditional r.
  Note that this is as if there is no information channel at all!

Task

$$\boxed{\begin{array}{ll} \text{maximize} & f(a_1,\ldots,a_n) = \sum_{s=1}^{n} q_s \log a_s \\ \text{subject to} & g(a_1,\ldots,a_n) = \sum_{s=1}^{n} a_s = 1 \end{array}}$$

"gambler bets entire capital" constraint.

↳ Lagrange Multipliers!

Treat $\vec{q}$ as known, and find the maximizing gambling strategy $\vec{a}$.

$$\vec{\nabla} f = \left(\frac{q_1}{a_1}, \ldots, \frac{q_n}{a_n}\right) \qquad \vec{\nabla} g = (1, \ldots, 1)$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \implies \left(\frac{q_1}{a_1}, \ldots, \frac{q_n}{a_n}\right) = \lambda(1, \ldots, 1)$$

$$\implies \lambda = \frac{q_1}{a_1} = \cdots = \frac{q_n}{a_n}$$

well, $\frac{q_1}{a_1} = \frac{q_s}{a_s} \implies a_s = a_1 \frac{q_s}{q_1} \quad \forall s$

Constraint $1 = \sum' a_s = \frac{a_1}{q_1}\left(\sum' q_s\right) = \frac{a_1}{q_1}$

$$\implies a_1 = q_1 \implies \boxed{a_s = q_s \quad \forall s}$$

We know this critical point is a maximizer because

$$\begin{cases} f(a_1 = q_1, \ldots, a_n = q_n) = \sum_k' q_k \log q_k \in \mathbb{R} \\ f(a_1 = 1, a_2 = 0, \ldots, a_n = 0) = -\infty \end{cases}$$

- Re-incorporating the received symbols r, we have found the gambling strategy that maximizes G, in the fair odds case:

$$\boxed{a(s|r) = q(s|r) = \frac{p(s,r)}{q(r)} = \frac{p(s,r)}{\sum_k' p(k,r)}}$$

Therefore,

$$G_{max} = \sum_{rs}' p(s,r) \log \underbrace{a(s|r)}_{= q(s|r)} - \sum_{rs}' p(s,r) \log p(s)$$

$$= -H(X|Y) + H(X)$$ "Shannon's Rate of Transmission"

2. case: Unfair odds, No Rake/Vigorish

No Rake $\boxed{\sum \frac{1}{\alpha_s} = 1}$

unfair odds α_s is not necessarily $\frac{1}{P(s)}$

Without loss of generality we still have $\boxed{1 = \sum_s a(s|r) \quad \forall r}$ since the gambler can still hold back money by betting in proportion to the $1/\alpha_s$, since no Rake.

As before,

$$G = \sum_{rs} P(s,r) \log\left[\alpha_s\, a(s|r)\right]$$

$$= \underbrace{\sum_{rs} P(s,r) \log a(s|r)}_{\text{To maximize } G \text{ is to maximize this}} + \underbrace{\sum_s P(s) \log \alpha_s}_{:= H(\alpha)}.$$

To maximize G is to maximize this, which we already know is when $\boxed{a(s|r) = q(s|r)}$. It is the same since this term doesn't depend on the odds α_s

$$G_{max} = H(\alpha) - H(X|Y)$$

Interesting Facts

(A.) G is maximized as before by setting $\boxed{a(s|r) = q(s|r)}$. So, the gambler ignores the posted odds in placing his bets!.

the better your information for an outcome, the more you should bet on that outcome.

(B.) Minimizing $H(\alpha) = \sum_s{}' p(s) \log \alpha_s$ subject to $\sum_s{}' \frac{1}{\alpha_s} = 1$ occurs when $\alpha_s = \frac{1}{p(s)}$, $H(X) = H(\alpha)$

(prove this using Lagrange multipliers).

So, any deviation from the fair odds helps the gambler!

this makes sense: the gambler can exploit unfair odds

Assume $\begin{cases} 60\% \text{ heads} \\ 40\% \text{ tails} \end{cases}$

odds $\begin{cases} 1/.7 \text{ heads} \\ 1/.3 \text{ tails} \end{cases}$

r constant

$$q(s|r) = q(s) = \begin{cases} .6 & H \\ .4 & T \end{cases}$$

Strategy $a(s) = \begin{cases} .6 \text{ of capital on } H \\ .4 \text{ of capital on } T \end{cases}$

$$G = \underbrace{\sum_{rs} p(s,r) \log a(s|r)}_{} + \underbrace{\sum_{s} p(s) \log o_s}_{}.$$

C.

$$G_{max} = H(\alpha) - H(X)$$ without information channel (ignore r)

$$G_{max} = H(\alpha) - H(X|Y)$$ with info

$$R = H(X) - H(X|Y)$$ "Shannon's Rate of Transmission"

= increase in G_{max} due to Information gain.

"If the odds are not fair (not consistent with the transmitted symbol probabilities but consistent with some other set of probabilities) then G_{max} is larger than it would have been with no Info Channel, by an amount equal to the Rate of Transmission of Information"