# Kelly Criterion. Case: Yes Rake/Vigorish/"Track Take"

Without a Rake, recall we wanted to maximize

$$G = \sum_{rs} p(s,r) \log\left[\alpha_s a(s|r)\right]$$

subject to $\sum_s' a(s|r) = 1.$

With a Rake, the constraint becomes

$$b_r + \sum_s' a(s|r) = 1,$$

$b_r > 0$ is the fraction of capital not bet, so we instead want to maximize

$$G = \sum_{rs}' p(s,r) \log\left[b_r + \alpha_s a(s|r)\right].$$

So,

$$\boxed{\begin{array}{l} \text{maximize } G = \sum_{rs}' p(s,r) \log\left[b_r + \alpha_s a(s|r)\right] \\ \text{subject to } b_r + \sum_s' a(s|r) = 1. \end{array}}$$

$$G = \sum_{rs}' p(s,r) \log\left[b_r + \alpha_s a(s|r)\right]$$

$$= \sum_r' p(r) \sum_s' p(s|r) \log\left[b_r + \alpha_s a(s|r)\right]$$

So to maximize $G$, we may maximize over each $r$ separately, which we can solve all at once by ignoring the information channel $r$.

$$\boxed{\begin{array}{l} \text{maximize } G = \sum_s' p(s) \log\left[b + \alpha_s a(s)\right] \\ \text{subject to } \quad b + \sum_s' a(s) = 1. \end{array}}$$

Need to choose $(a_1, .., a_n, b)$ that solves this, use notation $a_s$ instead of $a(s)$.

# KKT Conditions (as stated on Wikipedia)

**Optimization Problem**

minimize $f(\mathbf{x})$

subject to

$$g_i(\mathbf{x}) \leq 0,$$
$$h_j(\mathbf{x}) = 0.$$

**Lagrangian**

$$L(\mathbf{x}, \mu, \lambda) = f(\mathbf{x}) + \mu^\top \mathbf{g}(\mathbf{x}) + \lambda^\top \mathbf{h}(\mathbf{x})$$

**Theorem.** If $(\mathbf{x}^*, \mu^*)$ is a saddle point of $L(\mathbf{x}, \mu)$ in $\mathbf{x} \in \mathbf{X}$, $\mu \geq \mathbf{0}$, then $\mathbf{x}^*$ is an optimal vector for the above optimization problem.

- So, we need to write our optimization problem in terms of this KKT problem, and then solve $\nabla \mathcal{L} = 0$.

**Idea** Let $A = \{s : a_s > 0\}$, $A' = \{s : a_s = 0\}$.

**Let** $\vec{x} = (a_1, \dots, a_n, b)$

**Minimize** $f(\vec{x}) = -\sum_{s=1}^{n} P_s \log[b + d_s a_s]$

**Subject to**

$$\begin{cases} g_b(\vec{x}) = -b < 0 & \\ g_s(\vec{x}) = -a_s < 0 & \forall s \in A \\ h_s(\vec{x}) = -a_s = 0 & \forall s \in A' \\ H(\vec{x}) = b + \sum_{s=1}^{n} a_s - 1 = 0 & \end{cases}$$

Lagrangian

$$\boxed{\mathcal{L}(\vec{x},\vec{\mu},\vec{\lambda},K) = f(\vec{x}) + \vec{\mu}^T\vec{g}(\vec{x}) + \vec{\lambda}^T\vec{h}(x) + K\,H(\vec{x})}$$

Solve $\nabla\mathcal{L} = 0$.

$$\boxed{\begin{aligned}
&\forall s\in A, && \frac{\partial\mathcal{L}}{\partial a_s} = \frac{-p_s \alpha_s}{b+\alpha_s a_s}\log_2 e - \mu_s + K = 0\\
&\forall s\in A', && \frac{\partial\mathcal{L}}{\partial a_s} = \frac{-p_s \alpha_s}{b}\log_2 e - \lambda_s + K = 0\\
& && \frac{\partial\mathcal{L}}{\partial b} = -\sum_{s=1}^{n}\frac{p_s}{b+\alpha_s a_s}\log_2 e + K = 0
\end{aligned}}$$

Note $\forall s\in A$, $\mu_s = 0$, since if not, then $\frac{\partial\mathcal{L}}{\partial\mu_s} = -a_s < 0$, so $\frac{\partial\mathcal{L}}{\partial\mu_s} \neq 0$, which would contradict that $\nabla\mathcal{L} = 0$ at the optimal point.

Note $\forall s\in A'$, $\lambda_s \geq 0$ (a simplifying assumption ??)

Therefore, solve:

$$\frac{p(s)\alpha_s}{b+a(s)\alpha_s}\log e = k \qquad for\ s \epsilon \mathbf{A}$$

$$\sum_s \frac{p(s)}{b+a(s)\alpha_s}\log e = k$$

$$\frac{p(s)\alpha_s}{b}\log e \leqq k \qquad for\ s \epsilon \mathbf{A'}$$

Solve for $(a_1, \ldots, a_n, b)$.

---

<u>want</u> K to be constant.

<u>see</u> $\underbrace{K}_{\text{want constant.}} = \underbrace{\log_2 e}_{\text{constant.}} \; \underbrace{\frac{P_s \alpha_s}{b + a_s \alpha_s}}_{\text{choose } a_s \text{ to make this constant.}} \qquad \forall s \in A$

<u>try</u> $\boxed{a_s = P_s - \frac{b}{\alpha_s} \qquad \forall s \in A}$

$$\Rightarrow \log_2 e \left(\frac{P_s \alpha_s}{b + a_s \alpha_s}\right) = \log_2 e = K$$

$$\Rightarrow \boxed{K = \log_2 e}$$

- <u>Need</u> to solve for $b$, and the set $A$.

$$\log_2 e = K = \log_2 e \cdot \left( \sum_{s=1}^{n} \frac{p_s}{b + \alpha_s a_s} \right)$$

$$= \log_2 e \cdot \left( {\sum_{s \in A'}}' \frac{p_s}{b} + {\sum_{s \in A}}' \frac{1}{\alpha_s} \right)$$

$$\Longrightarrow \frac{1}{b} {\sum_{s \in A'}}' p_s = 1 - {\sum_{s \in A}}' \frac{1}{\alpha_s}$$

$$\Longrightarrow \frac{1}{b} \left( 1 - \sum_{s \in A} p_s \right) = 1 - {\sum_{s \in A}}' \frac{1}{\alpha_s}$$

$$\Longrightarrow \boxed{b = \frac{1 - \varphi}{1 - \sigma}} \quad \text{where} \quad \boxed{\begin{array}{l} \varphi = {\sum_{s \in A}}' p_s \\ \sigma = {\sum_{s \in A}}' \frac{1}{\alpha_s} \end{array}}$$

- Need to solve for the set $A$.

$$\forall s \in A', \quad \frac{p_s \alpha_s}{b} \log_2 e \le K = \log_2 e$$

$$\Longrightarrow \boxed{p_s \alpha_s \le b = \frac{1 - \varphi}{1 - \sigma} \quad \forall s \in A'}$$

Choose $A, A'$ based on this.

So, Choose $A$ based on

$$
\begin{aligned}
&\sigma < 1 \\
&p(s)\alpha_s \quad > \quad \frac{1-\varphi}{1-\sigma} \qquad \text{for } s \in A \\
&p(s)\alpha_s \quad \leqq \quad \frac{1-\varphi}{1-\sigma} \qquad \text{for } s \in A'
\end{aligned}
$$

- This process is summarized as follows:

(a) Permute indices so that $p(s)\alpha_s \geqq p(s+1)\alpha_{s+1}$

(b) Set b equal to the minimum positive value of

$$\frac{1-p_t}{1-\sigma_t} \quad \text{where} \quad p_t = \sum_1^t p(s), \;\; \sigma_t = \sum_1^t \frac{1}{\alpha_s}$$

(c) Set $a(s) = p(s) - b/\alpha_s$ or zero, whichever is larger. (The $a(s)$ will sum to $1-b$.)

$\square$

# Interesting Fact (Case: 3 teams, ($P_1+P_2<1$, $P_3>0$))

One of the results in the paper is that there exist horses with true probabilities and odds that make bets on the horse negative EV. Yet Kelly betting on the horse suggests wagering some amount of your bankroll anyway.

$$\begin{cases} P_1+P_2<1 \\ P_1, P_2 \geq 0 \end{cases}$$ probabilities

$$\begin{cases} P_1\alpha_1-1>0 \\ P_2\alpha_2-1<0 \end{cases}$$

option 1 is $+EV$ $\Rightarrow$ we will bet on some team

option 2 is $-EV$

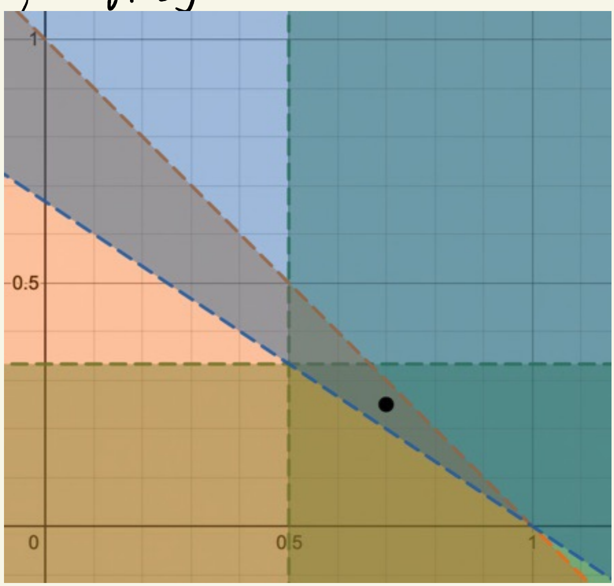
$$\mathrm{EV}(\$1\text{ bet}) = \binom{\text{profit}}{\text{amount}}\cdot\binom{P}{\text{win}} - \binom{\text{bet}}{\text{size}}\binom{P}{\text{lose}}$$
$$= (\alpha-1)P - 1(1-P)$$
$$= \alpha P - P - 1 + P$$
$$= P\alpha - 1$$

$$\begin{cases} 0 < \dfrac{1-P_1-P_2}{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}} < \dfrac{1-P_1}{1-\frac{1}{\alpha_1}} \\ \text{Equivalently, } 0 < F_2 < F_1 \end{cases}$$

make a nonzero bet on team 2!

$\alpha_1 \mapsto a$, $\alpha_2 \mapsto b$, $P \mapsto x$, $q \mapsto y$

1. $a = 2$ (slider −10 to 10)
2. $b = 3$ (slider −10 to 10)
3. $x \cdot a - 1 > 0$
4. $y \cdot b - 1 < 0$
5. $x + y < 1$
6. $0 < \dfrac{(1-x-y)}{\left(1-\frac{1}{a}-\frac{1}{b}\right)}$ → Kelly's "b"
7. $\dfrac{(1-x-y)}{\left(1-\frac{1}{a}-\frac{1}{b}\right)} < \dfrac{(1-x)}{\left(1-\frac{1}{a}\right)}$
8. $(.7,.25)$ Label




<u>Check</u> $a_2 = \max\left(P_2 - \frac{b}{\alpha_2}, 0\right) = 0.1$, so we make a $-EV$ bet as desired!