Horse Racing, Kelly Betting, and Unintuitive Bets

Ryan Brill

October 4, 2021

Decimal Odds

- Imagine you are betting on a horse race with, say, 3 horses. Robin will be one of the horses
- The *odds* are the prices for betting on each horse, set by the casino
- *Decimal odds*: $\alpha_{Robin} = 3$ means bet \$1 to profit \$2

EV

- EV = expected profit (value) of your bet
- Let $p_{Robin} = 0.3$ = true probability that Robin wins
- Let $\alpha_{Robin} = 3$ = decimal odds that Robin wins
- $\boxed{EV = p\alpha - 1}$ because

$$\begin{aligned} EV &= p \cdot (\text{profit if win bet}) + (1 - p) \cdot (\text{loss if lose bet}) = \\ &= p(\alpha - 1) + (1 - p)(-1) \\ &= p\alpha - 1 \end{aligned}$$

- $EV_{Robin} = -0.1$, sad!

Food for Thought

- Does there exist a scenario in which you would make a Negative EV bet?

Food for Thought 2

- Does there exist a scenario in which you would make a Negative EV bet *in the long run*?

Money Function

- Suppose we bet on the same horse race for N consecutive horse races.
- n horses
- Decimal odds $(\alpha_1, ..., \alpha_n)$
- True win probabilities $(p_1, ..., p_n)$
- Fraction of your money $(x_1, ..., x_n)$ bet on each horse
- Fraction of your money b not bet
- Horse index $h \in \{1, ..., n\}$
- Race index $r \in \{1, ..., N\}$
- $W_{rh} = 1$ if horse h Wins race r, else 0. $W_{rh} \sim \text{Bernoulli}(p_h)$
- $V_N =$ capital after N races, starting with \$1

$$\boxed{V_N = \prod_{r,h} \left[b + x_h \alpha_h \right]^{W_{rh}}}$$

Money Function

$$\boxed{V_N = \prod_{r,h} \left[b + x_h \alpha_h\right]^{W_{rh}}}$$

- Want to choose $(b, x_1, ..., x_n)$ to maximize $\mathbb{E} V_N$
- *Problem*: Maximizing the expectation of a product of random variables is difficult

Kelly and Jensen

$$V_N = \prod_{r,h} \left[b + x_h \alpha_h \right]^{W_{rh}}$$

- *Idea*: take the log
- $\log \mathbb{E} V_N$ doesn't help
- *Kelly*: Instead, maximize $\mathbb{E} \log V_N$
- *Jensen*: $\log \mathbb{E} V_N \geq \mathbb{E} \log V_N$
- Sometimes, $\log \mathbb{E} V_N > \mathbb{E} \log V_N$

Optimization

$$\begin{aligned} &\text{argmax}_{(b,x_1,\ldots,x_n)}\ \mathbb{E} \log V_N \\ = &\text{argmax}_{(b,x_1,\ldots,x_n)}\ \mathbb{E} \sum_{r,h} W_{rh} \log(b + x_h \alpha_h) \\ &\text{subject to} \quad b + \sum_h x_h = 1 \end{aligned}$$

- Doable!
- ...Lagrange multipliers / KKT conditions...
- Yields a formula for $(x_1, \ldots, x_n)$, the fraction of your captial that you should bet on each horse!

Ugly formula for $(x_1, ..., x_n)$

1. Permute indices so that $p_h\alpha_h > ... > p_{h+1}\alpha_{h+1}$ (arrange the horses by decreasing order of EV)
2. The fraction of your capital that you don't bet, b, is the minimum positive value of

   $$\frac{1 - \sum_{h=1}^{t} p_h}{1 - \sum_{h=1}^{t} \frac{1}{\alpha_h}}$$

   for $t \in \{1, ..., n\}$
3. Set

   $$x_h = \max\{p_h - \frac{b}{\alpha_h}, \;\; 0\}$$

Food for Thought 3

- Does there exist a scenario in which you would make a Negative EV bet *in the long run*?
- According to Kelly, *yes*

Example of a Kelly-Optimal, Negative EV Bet

- 3 horses
- True win probabilities $p = (.6, .3, .1)$
- Decimal odds $\alpha = (2, 3, 8)$
- Kelly says

$$\begin{cases} b = 0.6, \\ x_1 = 0.3, \\ x_2 = 0.1, \\ x_3 = 0 \end{cases}$$

- Bet $1/10^{th}$ of our captial on horse 2 (Robin), which is a Negative EV bet:

$$EV_2 = p_2\alpha_2 - 1 = (.3)3 - 1 = -0.1$$

- Why?

Why make a Negative EV Bet?

- Because using the *compound capital*

$$V_N = \prod_{r,h} \left[b + x_h \alpha_h \right]^{W_{rh}}$$

- You're compounding your money and don't want to go bankrupt
- If we instead maximize one race by itself, we would put all our money on the horse with the highest EV

Case: Fair Odds

- If the odds are *fair*

$$\sum \frac{1}{\alpha_h} = 1$$

then

$$x_h = p_h$$

- Our allocations don't depend on the odds at all!

End

- Please help me get true win probabilities $(p_1, ..., p_n)$